Onoochin's Paradox

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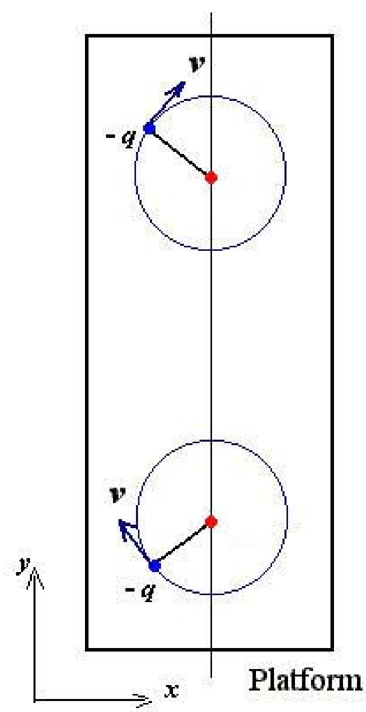
1 The Paradox

Vladimir Onoochin has drawn our attention to a paradox related to the apparatus shown in the figure below. Two electric charges $-q$ are forced to execute uniform circular motion with speed v such that their azimuthal angles are always 90° apart. The apparatus, including the motors which drive the motion, is mounted on a frictionless platform such that there are no external forces on the system that have components in the horizontal (x-y) plane. Therfore, we expect that the center of mass of the system would remain at rest. However, the magnetic Lorentz forces of the moving charges on the each other are not equal and opposite, so the sum of the internal forces is nonzero. In particular, the total (magnetic) Lorentz force has a component in the $-x$ direction at all times during the motion. Hence, we are led to conclude that the center of mass of this system is accelerated monotonicially in the $-x$ direction by its own internal forces, in contradiction to the principles of Newtonian mechanics.



The sum of the (magnetic) Lorentz forces on the two charges is (in Gaussian units) of order q^2v^2/c^2r^2, where c is the speed of light and r is the average distance between the two charges. For any reasonable tabletop apparatus the apparently offending Lorentz force is extremely tiny. Thus, the issue of primarily one of principle: Are the laws of electromagnetism are compatible with the laws of mechanics?

2 Ampère, Biot, Savart and Lorentz

Concerns such as those raised by Onoochin appear to have influenced Ampère in 1822 [1] when he wrote the force between two circuits that carry currents I_1 and I_2 as

$$\mathbf{F} = \frac{I_1 I_2}{c^2} \oint \oint \left[d\mathbf{l}_1 \cdot d\mathbf{l}_2 - \frac{3}{2}(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) \right] \frac{\hat{\mathbf{r}}}{r^2} \qquad \text{(Ampère)}, \tag{1}$$

in preference to the (now) more familiar Biot-Savart law (as later expressed by Grassmann [3]),

$$\mathbf{F} = \frac{I_1 I_2}{c^2} \oint \oint d\mathbf{l}_1 \times \frac{d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2} \qquad \text{(Biot-Savart-Grassmann)}, \tag{2}$$

because the integrand of eq. (1), but not that of eq. (2), lies along the line joining current elements in the two circuits, and implies that the forces of a pair of current elements on each other are equal and opposite. However, as noted by Ampére in 1826 [4], the Biot-Savart law (2) is more compatible with the concept of the magnetic field,

$$\mathbf{F} = \oint \frac{I_1 d\mathbf{l}_1}{c} \times \mathbf{B}, \qquad \mathbf{B} = \oint \frac{I_2 d\mathbf{l}_2}{c} \times \frac{\hat{\mathbf{r}}}{r^2} \qquad \text{(Ampère-Biot-Savart)}. \tag{3}$$

The magnetic term in the Lorentz force law [5] follows from eq. (3) upon replacement of a current element $I d\mathbf{l}$ by the product $q\mathbf{v}$ of an electric charge q and its velocity $\mathbf{v}$,

$$\mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{B} = q\frac{\mathbf{v}}{c} \times \frac{\hat{\mathbf{r}}}{r^2} \qquad \text{(Lorentz)}. \tag{4}$$

As the extrapolation (4) of the Biot-Savart law to isolated current elements leads to apparent contradictions with Newton's laws of mechanics, Ampère seems to have concluded that isolated current elements could not exist.

Maxwell did little to change this view of Ampère, since he considered electric charge to be a continuously distributed density ρ related to strain in an æther according to $\rho = \boldsymbol{\nabla} \cdot \mathbf{D}/4\pi$, where $\mathbf{D}$ is the electric displacement field.

An electrodynamics based on charged particles could be developed in a manner consistent with Newtonian mechanics only after the efforts of Thomson [6] and Poynting [7] who showed that electromagnetic fields can carry energy and momentum.[1] The Lorentz force on a charged particle was first discussed by Thomson in sec. 5 of [6]. For a review by Lorentz, see [10].

However, the doubts of Ampère as to the compatibility of the Lorentz force law with Newton's laws are seldom addressed explicitly in the electrodynamic theory of charged particles, so that examples such as that of Onoochin may well appear paradoxical.

3 The Lorentz Force of a Pair of Charges to Order $1/c^2$

A paper by Page and Adams [11] provides good insight as to the resolution of Onoochin's paradox: at order $1/c^2$ both the electric and the magnetic fields of a pair of charges contribute to the nonzero total Lorentz force. For another viewpoint, see Appendix B.

[1]That a moving charge interacting with thermal radiation should feel a radiation pressure was anticipated by Stewart in 1871-3 [8], who inferred that both the energy and the momentum of the charge would be affected. In 1873, Maxwell discussed the pressure of light on conducting media at rest, and on "the medium in which waves are propagated" ([9], secs. 792-793).

For calculations of the Lorentz force to be accurate to order $1/c^2$, it suffices to use eq. (4) for the magnetic field. However, to maintain the desired accuracy the electric field of a moving charge must include effects of retardation, as can be obtained from an expansion of the Liénard-Wiechert fields [12, 13] (for details, see the Appendix of [14]),

$$\mathbf{E} \approx q\frac{\hat{\mathbf{r}}}{r^2}\left(1 + \frac{v^2}{2c^2} - 3\frac{(\hat{\mathbf{r}}\cdot\mathbf{v})^2}{2c^2}\right) - \frac{q}{2c^2 r}[\mathbf{a} + (\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}], \tag{5}$$

where $\mathbf{a}$ is the acceleration of the charge at the present time.[2] Then, the total electromagnetic force $\mathbf{F}_{\rm EM}$ on a pair of accelerating charges separated by distance $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is, to order $1/c^2$,

$$\begin{aligned}\mathbf{F}_{\rm EM} \;=\; -\frac{q_1 q_2}{2c^2 r}\Big\{ & \mathbf{a}_1 + \mathbf{a}_2 + [(\mathbf{a}_1 + \mathbf{a}_2)\cdot\hat{\mathbf{r}}]\hat{\mathbf{r}} - \frac{\hat{\mathbf{r}}}{r}\left[v_1^2 - v_2^2 - 3(\hat{\mathbf{r}}\cdot\mathbf{v}_1)^2 + 3(\hat{\mathbf{r}}\cdot\mathbf{v}_2)^2\right] \\ & - \frac{2\hat{\mathbf{r}}}{r}\times(\mathbf{v}_1\times\mathbf{v}_2)\Big\},\end{aligned} \tag{6}$$

where the triple cross product describes the effect of the magnetic field.

Onoochin's example involves uniform circular motion, $v_1 = v_2 = v$ and $a_1 = a_2 = v^2/R$, where the radius R of the circle is smaller than the distance r between the two charges. Hence, the largest contribution to the total electromagnetic force is from the terms in the electric fields involving the acceleration. It turns out that some of these terms exactly cancel the force due to the magnetic fields, thereby resolving Onoochin's paradox.

It may be useful to evaluate eq. (6) in greater detail for Onoochin's example in the case that the separation between the centers of the two rotating disks is $D\,\hat{\mathbf{y}}$ where $R \ll D$. We will evaluate the force (6) to order R/D.

The positions, velocities and accelerations of the two charges can be written as

$$\mathbf{r}_1 = R\sin\frac{vt}{R}\,\hat{\mathbf{x}} + R\cos\frac{vt}{R}\,\hat{\mathbf{y}}, \qquad \mathbf{r}_2 = D\,\hat{\mathbf{y}} + R\cos\frac{vt}{R}\,\hat{\mathbf{x}} - R\sin\frac{vt}{R}\,\hat{\mathbf{y}}, \tag{7}$$

$$\mathbf{v}_1 = v\cos\frac{vt}{R}\,\hat{\mathbf{x}} - v\sin\frac{vt}{R}\,\hat{\mathbf{y}}, \qquad \mathbf{v}_2 = -v\sin\frac{vt}{R}\,\hat{\mathbf{x}} - v\cos\frac{vt}{R}\,\hat{\mathbf{y}}, \tag{8}$$

$$\mathbf{a}_1 = -\frac{v^2}{R}\sin\frac{vt}{R}\,\hat{\mathbf{x}} - \frac{v^2}{R}\cos\frac{vt}{R}\,\hat{\mathbf{y}}, \qquad \mathbf{a}_2 = -\frac{v^2}{R}\cos\frac{vt}{R}\,\hat{\mathbf{x}} + \frac{v^2}{R}\sin\frac{vt}{R}\,\hat{\mathbf{y}}. \tag{9}$$

For this example, the terms in eq. (6) involving the velocities are of order R/D times the leading terms in the accelerations. Hence, to maintain accuracy to order R/D it suffices to approximate r by D and $\hat{\mathbf{r}}$ by $\hat{\mathbf{y}}$ in the terms involving the velocities. However, we must keep the first corrections to r and $\hat{\mathbf{r}}$ in the terms involving the accelerations. Therefore, we record the relations:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = D\left[\hat{\mathbf{y}} + \frac{R}{D}\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\hat{\mathbf{x}} - \frac{R}{D}\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right)\hat{\mathbf{y}}\right], \tag{10}$$

[2] Expression (5) may be counterintuitive in that a more usual expression for the part of the electric field that varies as $1/r$ is $-q[\mathbf{a}_\perp/c^2 r]_{\rm ret} = -q[(\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})/c^2 r]_{\rm ret}$, with acceleration $\mathbf{a}$ evaluated at the retarded time $t' = t - r/c$. Nonetheless, expression (5) is the result of the conversion to present quantities of the usual form that involves retarded quantities. Note that the Poynting vector deduced from eqs. (4) and (5) varies as $1/r^3$, so that the phenomenon of radiation does not appear in the present approximation. Radiation appears only when higher-order terms are considered [15].

$$\frac{1}{r} = \frac{1}{D\sqrt{1 - \frac{2R}{D}\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right) + \frac{2R^2}{D^2}}} \approx \frac{1}{D}\left[1 + \frac{R}{D}\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right)\right], \tag{11}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \approx \hat{\mathbf{y}} + \frac{R}{D}\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\hat{\mathbf{x}}, \tag{12}$$

$$\begin{aligned} \frac{\mathbf{a}_1 + \mathbf{a}_2 + [(\mathbf{a}_1 + \mathbf{a}_2)\cdot\hat{\mathbf{r}}]\hat{\mathbf{r}}}{r} \approx & -\frac{v^2}{DR}\left[\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right)\hat{\mathbf{x}} + 2\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\hat{\mathbf{y}} \right. \\ & \left. + \frac{2R}{D}\,\hat{\mathbf{x}} + \frac{3R}{D}\cos\frac{2vt}{R}\,\hat{\mathbf{y}}\right], \end{aligned} \tag{13}$$

$$\frac{\hat{\mathbf{r}}}{r^2}\left[v_1^2 - v_2^2 - 3(\hat{\mathbf{r}}\cdot\mathbf{v}_1)^2 + 3(\hat{\mathbf{r}}\cdot\mathbf{v}_2)^2\right] \approx -\frac{3v^2}{D^2}\cos\frac{2vt}{R}\,\hat{\mathbf{y}}, \tag{14}$$

$$\frac{2\hat{\mathbf{r}}}{r^2}\times(\mathbf{v}_1\times\mathbf{v}_2) \approx -\frac{2v^2}{D^2}\,\hat{\mathbf{x}}, \tag{15}$$

and we find that equation (6) can be approximated as

$$\mathbf{F}_{\rm EM} \approx \frac{q_1 q_2}{2DR}\frac{v^2}{c^2}\left[\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right)\hat{\mathbf{x}} + 2\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\hat{\mathbf{y}} + \frac{6R}{D}\cos\frac{2vt}{R}\,\hat{\mathbf{y}}\right]. \tag{16}$$

The magnetic force term (15) is canceled by a correction of order R/D to the much larger part of the electric force associated with the acceleration of the charges, as seen in eq. (13). Thus, the most dramatic aspect of Onoochin's paradox is resolved; the total electromagnetic force is purely oscillatory. However, the net electromagnetic force (16) on the (isolated) system is nonzero, so there remains the paradox of Ampère that Newton's third law is not obeyed by the Lorentz force between pairs of moving charges.

4 Electromagnetic Field Momentum

Following Poynting [7],[3] we now consider the momentum in the electromagnetic fields, calculated according to

$$\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\; d\mathrm{Vol}. \tag{17}$$

However, we cannot distinguish the "self-momentum" $\int \mathbf{E}_j \times \mathbf{B}_j\; d\mathrm{Vol}/4\pi c$ from the mechanical momentum of particle j.[4] Rather, we need to consider the interaction momentum

$$\mathbf{P}_{\rm field} = \frac{1}{4\pi c}\int (\mathbf{E}_1\times\mathbf{B}_2 + \mathbf{E}_2\times\mathbf{B}_1)\; d\mathrm{Vol}. \tag{18}$$

[3]O. Heaviside independently invented in 1885 [16] what is now called the Poynting vector, following his invention in 1882 of modern vector notation [17]. The dual role of the Poynting vector as both electromagnetic energy flux and electromagnetic momentum density was first pointed out by Abraham [18].

[4]To order $1/c^2$, the mechanical momentum of a particle of (rest) mass m and velocity $\mathbf{v}$ is $\mathbf{P}_{\rm mech} = m\mathbf{v}(1 + v^2/2c^2)$. Use of this form in the equation of motion (20) includes the effects of the electromagnetic "self-momentum".

Keeping terms only of order $1/c^2$, the relevant electromagnetic momentum is [11]

$$\mathbf{P}_{\rm field} = \frac{q_1 q_2}{4\pi c^2} \int \frac{\hat{\mathbf{r}}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}}_1) + \hat{\mathbf{r}}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{r}}_2)}{r_1^2 r_2^2}\, d{\rm Vol} = \frac{q_1 q_2}{2c^2 r}[\mathbf{v}_1 + \mathbf{v}_2 + ((\mathbf{v}_1 + \mathbf{v}_2) \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}]. \quad (19)$$

While strictly speaking the electromagnetic momentum is a property of the system as a whole, the terms in eq. (19) containing $\mathbf{v}_j$ can be identified as the interaction momentum associated with particle j.

Taking the time derivative of eq. (19), we find that

$$\frac{d\mathbf{P}_{\rm field}}{dt} = -\mathbf{F}_{\rm EM}, \quad (20)$$

on comparing with eq. (6). Thus, a calculation of the electromagnetic momentum according to eqs. (18)-(19) provides us with a solution to the equation of motion (20).[5]

For related discussion of some additional examples, see [19].

5 Two Interacting But Otherwise Free Charges

If the only forces on the two charges are their electromagnetic forces on each other, then Newton's second law tells us that

$$\frac{d\mathbf{P}_{\rm mech}}{dt} = \mathbf{F}_{\rm EM}, \quad (22)$$

where $\mathbf{P}_{\rm mech}$ is the combined mechanical momentum of the two particles. Then, with eq. (20) we have

$$\frac{d\mathbf{P}_{\rm total}}{dt} = \frac{d\mathbf{P}_{\rm mech}}{dt} + \frac{d\mathbf{P}_{\rm field}}{dt} = 0. \quad (23)$$

In this case the center of mass of the two particles can move as a result of their Lorentz forces on each other, but the center of momentum of the system remains fixed (or in a state of uniform motion).

6 Resolution of Onoochin's Paradox

Turning to Onoochin's example in which the two charges undergo uniform circular motion, there are other forces on the charges besides the Lorentz force. The motion, and hence also

[5]This result should not come as a surprise in that one version of Poynting's argument transforms the Lorentz force $\mathbf{F}_{\rm EM}$ on a system of charges according to

$$\mathbf{F}_{\rm EM} = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{Area} - \frac{1}{4\pi c}\frac{d}{dt}\int \mathbf{E} \times \mathbf{B}\, d{\rm Vol}, \quad (21)$$

where $\overleftrightarrow{\mathbf{T}}$ is the Maxwell stress tensor. See, for example, sec. 10 of [10]. For a bounded system of charges *for which radiation can be neglected* the integral of the stress tensor vanishes as the surface/volume of integration grow large, leading to eq. (20). Radiation can be neglected in the present analysis, which is accurate to order $1/c^2$, because radiation effects are of order $1/c^3$ according to the Larmor formula, $dU/dt = 2q^2a^2/3c^3$.

the mechanical momentum $\mathbf{P}_{\rm mech}$ of the charges, its rate of change $d\mathbf{P}_{\rm mech}/dt$, and the total force $\mathbf{F}_{\rm total}$ on the charges are known (to a first approximation), so we write

$$\frac{d\mathbf{P}_{\rm mech}}{dt} = \mathbf{F}_{\rm total} = \mathbf{F}_{\rm EM} + \mathbf{F}_{\rm other}. \tag{24}$$

We should also consider the possible motion of the (frictionless) supporting platform, which is subject to the reaction force $-\mathbf{F}_{\rm other}$, assuming that the forces between the platform and the charges obey Newton's 3rd law. The equation of motion of the platform can therefore be written as

$$\frac{d\mathbf{P}_{\rm platform}}{dt} = -\mathbf{F}_{\rm other}. \tag{25}$$

The combined equation of motion for the platform plus the charges follows from eqs. (24)-(25) and eq. (20) as

$$\frac{d\mathbf{P}_{\rm mech}}{dt} + \frac{d\mathbf{P}_{\rm platform}}{dt} = \mathbf{F}_{\rm EM} = -\frac{d\mathbf{P}_{\rm field}}{dt}\,. \tag{26}$$

Hence, the total momentum of the system, $\mathbf{P}_{\rm field} + \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm platform}$, is constant as expected.

For the example of Onoochin's paradox with velocities described by eq. (8) the field momentum (19) is (recalling eqs. (11)-(12) and calculating to order $1/c^2$ and to order R/D),

$$\begin{aligned} \mathbf{P}_{\rm field} &= -\mathbf{P}_{\rm mech} - \mathbf{P}_{\rm platform} \\ &\approx \frac{q_1 q_2 v}{2c^2 D}\left[\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\hat{\mathbf{x}} - 2\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right)\hat{\mathbf{y}} - \frac{R}{D}\left(1 + 3\sin\frac{2vt}{R}\right)\hat{\mathbf{y}}\right]. \end{aligned} \tag{27}$$

We readily see that the time derivative of eq. (27) is the negative of the force $\mathbf{F}_{\rm EM}$ found in eq. (16).

However, the presence of the constant term, $-(q_1 q_2 v R/2c^2 D^2)\,\hat{\mathbf{y}}$, in eq. (27) is surprising. This suggests that there is a kind of "hidden" mechanical momentum in the system that is equal and opposite to this constant term.

When the total energy of the system is constant, the electrostatic energy $q_1 q_2/r$ of a pair of charges q_1 and q_2 must be compensated by a change in their effective mechanical masses of $-q_1 q_2/2c^2 r$ for each charge. This tiny change is a relativistic effect of order $1/c^2$. If the charges have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ their mechanical momentum is changed by amount $-q_1 q_2(\mathbf{v}_1 + \mathbf{v}_2)/2c^2 r$. This small difference is sometimes called the "hidden" mechanical momentum $\mathbf{P}_h$ of the system [20, 21, 22].[6]

In the present example, we calculate the "hidden" momentum using eqs. (8), (11) and (12),

$$\begin{aligned} \mathbf{P}_h &= -\frac{q_1 q_2}{2c^2 r}(\mathbf{v}_1 + \mathbf{v}_2) \\ &\approx -\frac{q_1 q_2 v}{2c^2 D}\left[1 + \frac{R}{D}\left(\cos\frac{vt}{R} + \sin\frac{vt}{R}\right)\right]\left[\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\,\hat{\mathbf{x}} - \left(\cos\frac{vt}{R}\ + \sin\frac{vt}{R}\right)\,\hat{\mathbf{y}}\right] \\ &= -\frac{q_1 q_2 v}{2c^2 D}\left\{\left(\cos\frac{vt}{R} - \sin\frac{vt}{R}\right)\,\hat{\mathbf{x}} - \left(\cos\frac{vt}{R}\ + \sin\frac{vt}{R}\right)\,\hat{\mathbf{y}} \right. \\ &\qquad \left. + \frac{R}{D}\left[\cos\frac{2vt}{R}\,\hat{\mathbf{x}} - \left(1 + \sin\frac{2vt}{R}\right)\,\hat{\mathbf{y}}\right]\right\} \end{aligned} \tag{28}$$

[6] If the moving charges comprise a steady current $\mathbf{J}$ whose divergence is zero, the "hidden" mechanical momentum can be evaluated as $\mathbf{P}_h = -\int \Phi\mathbf{J}\, d{\rm Vol}/c^2$, where Φ is the electric potential [23, 24, 25].

The constant term in the "hidden" mechanical momentum (28) is indeed equal and opposite to the constant term in the field momentum (27).

It is perhaps noteworthy that the "hidden" mechanical momentum in this example is not simply the negative of the field momentum, although the total momentum is zero. The "hidden" mechanical momentum has the form $\mathbf{P}_h = -\int \phi\, \mathbf{J}\, d\text{Vol}$, *while the electromagnetic momentum can be rewritten (in the low-velocity limit) as* $\mathbf{P}_{\text{field}} = \int \phi\, \mathbf{J}_t\, d\text{Vol}$ *where the so-called transverse current* $\mathbf{J}_t$ *is that part of the total current that obeys* $\boldsymbol{\nabla} \cdot \mathbf{J}_t = 0$ *[26]. In examples with steady currents the condition* $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ *holds and* $\mathbf{P}_h = -\mathbf{P}_{\text{field}}$. *Here, the two charges execute periodic motion, but the corresponding currents are not divergence free and so* $\mathbf{P}_h \neq -\mathbf{P}_{\text{field}}$.

The "visible" momentum, $\mathbf{P}_v = m_{\text{system}} \mathbf{v}_{\text{CM}}$, of the disks + platform does not include the "hidden" momentum (28), and is given by

$$\begin{aligned} \mathbf{P}_v &= m_{\text{system}} \frac{d\mathbf{r}_{\text{CM}}}{dt} = \mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{platform}} - \mathbf{P}_h = -\mathbf{P}_{\text{field}} - \mathbf{P}_h \\ &= \frac{q_1 q_2 v}{2c^2 D} \left\{ \left(\cos \frac{vt}{R} + \sin \frac{vt}{R} \right) \hat{\mathbf{y}} + \frac{R}{D} \left[\cos \frac{2vt}{R}\, \hat{\mathbf{x}} + 2 \sin \frac{2vt}{R}\, \hat{\mathbf{y}} \right] \right\}. \end{aligned} \tag{29}$$

The center-of-mass motion $\mathbf{r}_{\text{CM}}(t)$ of the system can be obtained by integration of eq. (29), which yields

$$\mathbf{r}_{\text{CM}} = \frac{q_1 q_2}{m_{\text{system}} c^2} \left[\frac{R}{2D} \left(\sin \frac{vt}{R} - \cos \frac{vt}{R} \right) \hat{\mathbf{y}} + \frac{R^2}{4D^2} \left(\sin \frac{2vt}{R}\, \hat{\mathbf{x}} - 2 \cos \frac{2vt}{R}\, \hat{\mathbf{y}} \right) \right]. \tag{30}$$

This motion is periodic but somewhat complicated, with an extremely small amplitude.

7 Appendix A: Electrical Interactions of Circuits

Does the force between two current-carrying circuits include terms due to the retarded electric fields of the moving charges that are of similar size to the magnetic force? If so, Ampère's analysis of magnetism would be incorrect.

An important difference between a current-carrying wire and an isolated moving charge is that the wire is electrically neutral (in the first approximation). So, there is no net Coulomb electric field due to the charges in the wire. However, there is a nonzero electric field associated with the wire because only the moving charges generate the retarded corrections of order $1/c^2$ to their Coulomb fields. But, this nonzero electric field produces no net force on a second circuit if that circuit is also electrically neutral.

Are current-carrying wires actually electrically neutral?[7] If the wire has nonzero electrical conductivity σ, and carries current density $\mathbf{J}$, then there must be a longitudinal electric field $\mathbf{E}$ inside the wire according to Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$. This longitudinal electric field is created and shaped by the presence of a nonzero surface charge density whose magnitude varies

[7] Most discussions of the net charge of a current-carrying wire focus on a very small effect. Namely, that the volume charge density of the moving electrons must be $1 + v^2/c^2$ times that of the fixed positive charges, so that the conduction electrons experience no net radial force. However, the net charge density due to this effect is v/c times smaller than the surface charge density required to shape the longitudinal field inside the wire.

linearly along the wire (so that the magnitude of $\mathbf{E}$ is independent of position). The charge density per unit length is approximately $I\mathcal{R}z$ [25], where $\mathcal{R}$ is the resistance per unit length along the wire. The numerical value of $\mathcal{R}$ is a fraction of an Ohm per cm. Recall that in Gaussian units, $1/c = 30$ Ohm. Hence, the net charge per unit length of a current-carrying wire is of order I/c. The surface charge distribution has a dipole character, so its electric field outside the wire falls off as R/r^3 rather than $1/r^2$, where R is a characteristic radius of the circuits.

Hence, the electrical force between a pair of circuits separated by distance r due to their surface charge distributions scales as $I_1 I_2 R^3/c^2 r^3$ and can be neglected compared to the magnetic forces that scale as $I_1 I_2 R^2/c^2 r^2$ according to Ampère.

There also exist electrical forces between current-carrying circuits of order $I_1 I_2 v R^2/c^3 r^2$, being the product of a net charge of order $I_1 R/c$ in one circuit times the retarded correction of order $I_2 v R/c^2 r^2$ to the electric field of the moving charges in the other circuit. This correction to Ampère's analysis is of order v/c compared to the magnetic force, and hence is negligible for ordinary circuits.

8 Appendix B: Resolution via the Darwin Lagrangian

An alternative (and more formal) approach to that given in secs. 3-4 can be based on the approximate Lagrangian of Darwin [27] (see also sec. 65 of [28] and sec. 12.6 of [29]), which describes the interaction of charged particles via their electromagnetic potentials in the Coulomb gauge, accurate to order $1/c^2$.

Here, it suffices to note that the vector potential $\mathbf{A}_1$ of an electric charge q_1 with velocity v_1 at distance r is (to order $1/c^2$ in the Coulomb gauge)

$$\mathbf{A}_1 = \frac{q_1}{2cr}[\mathbf{v}_1 + (\mathbf{v}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}]. \tag{31}$$

The canonical momentum $\mathbf{p}_2$ of a charge q_2 in the field of charge 1 is then

$$\mathbf{p}_2 = \mathbf{P}_{\mathrm{mech},2} + q_2 \frac{\mathbf{A}_1}{c}\,, \tag{32}$$

where the term $q_2 \mathbf{A}_1/c$ is often called the electromagnetic momentum of charge 2.[8]

The electromagnetic field momentum of the combined system of charges q_1 and q_2 is therefore,

$$\mathbf{P}_{\mathrm{field}} = \frac{q_1 q_2}{2c^2 r}[\mathbf{v}_1 + \mathbf{v}_2 + (\mathbf{v}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} + (\mathbf{v}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}], \tag{33}$$

as found previously in eq. (19).

The total canonical momentum of an isolated system is, of course, constant in time, which again leads to the conclusions of secs. 5 and 6.

[8]Following Maxwell, who refers to the $\mathbf{A}/c$ as the "electrokinetic momentum" in sec. 604 of [9], without the modern qualifier "per unit charge".

9 References

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This meeting featured an inspirational address by W. Thomson as a memorial to Herschel; among many other topics Thomson speculates on the size of atoms, on the origin of life on Earth as due to primitive organisms arriving in meteorites, and on how the Sun's source of energy cannot be an influx of matter as might, however, explain the small advance of the perihelion of Mercury measured by LeVerrier.
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Stewart argued that the radiation resistance felt by a charge moving through blackbody radiation should vanish as the temperature of the bath went to zero, just as he expected the electrical resistance of a conductor to vanish at zero temperature.
The 43rd meeting was also the occasion of reports by Maxwell on the exponential atmosphere as an example of statistical mechanics (pp. 29-32), by Rayleigh on the diffraction limit to the sharpness of spectral lines (p. 39), and (perhaps of greatest

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